k-Conflict-Free Coloring and k-Strong-Conflict-Free Coloring for One Class of Hypergraphs and Online k-Conflict-Free Coloring

Zhen Cui and Ze-Chun Hu[1]

Department of Mathematics, Nanjing University, China
(e-mail: cuizhen.nju@gmail.com, huzc@nju.edu.cn)

Abstract Conflict-free coloring is a kind of coloring of hypergraphs requiring each hyperedge to have a color which appears only once. More generally, there are k-conflict-free coloring (k-CF-coloring for short) and k-strong-conflict-free coloring (k-SCF-coloring for short) for some positive integer k. Let H_n be the hypergraph induced by the points $\{1, 2, \ldots, n\}$ with respect to intervals. At first, we study the k-SCF-coloring of H_n and give the exact k-SCF-coloring number of H_n for $k = 2, 3$. Second, we give the exact k-CF-coloring number of H_n for all k. Finally, we extend some results about online conflict-free coloring for hypergraphs obtained in [5] to online k-CF-coloring.

1 Introduction

A *hypergraph* is a pair $(V, \mathcal{E})$ where V is a set and $\mathcal{E}$ is a collection of subsets of V. The elements of V are called *vertices* and the elements of $\mathcal{E}$ are called *hyperedges*. If for any $e \in \mathcal{E}$, $|e| = 2$, then the pair $(V, \mathcal{E})$ is a *simple graph*. For a subset $V' \subset V$, we call the hypergraph $H(V') = (V', \{S \cap V' | S \in \mathcal{E}\})$ the *sub-hypergraph* induced by V'. An *m-coloring* for some $m \in \mathbb{N}$ of (the vertices of) H is a function $\phi : V \to \{1, \ldots, m\}$. Let ϕ be an *m-coloring* of H, if for any $e \in \mathcal{E}$ with $|e| \geq 2$, there exist at least two vertices $x, y \in e$ such that $\phi(x) \neq \phi(y)$, we call ϕ *proper* or *non-monochromatic*. Let $\chi(H)$ denote the least integer m for which H admits a proper coloring with m colors. The following coloring is more restrictive than non-monochromatic coloring.

Definition 1.1 (Conflict-Free Coloring) *Let $H = (V, \mathcal{E})$ be a hypergraph and let $C : V \to \{1, \ldots, m\}$ be some coloring of H. C is called a conflict-free coloring (CF-coloring for short) if for any $e \in \mathcal{E}$ there is a vertex $x \in e$ such that $\forall y \in e, y \neq x \Rightarrow C(y) \neq C(x)$.*

Let $\chi_{cf}(H)$ denote the least integer m for which H admits a CF-coloring with m colors.

The notion of CF-coloring was first introduced and studied by Smorodinsky [19] and Even et al. [12]. Such coloring is very useful in wireless networks, radio frequency identification

(RFID) networks and vertex ranking prolem. Refer to the survey paper [21] by Smorodinsky and the references therein for more descriptions. Such coloring have attracted many researchers from the computer science and mathematics community. As to CF-coloring of hypergraphs that arise in geometry, refer to Smorodinsky [19], Even et al. [12], Har-Peled and Smorodinsky [14], Smorodinsky [20], Pach and Tardos [18], Ajwani et al. [2], Chen et al. [10], Alon and Smorodinsky [3], Lev-Tov and Peleg [16] and etc. As to CF-coloring of arbitrary hypergraphs, refer to Pach and Tardos [17].

Smorodinsky [19] considered extensions of CF-coloring and introduced the following notion.

Definition 1.2 (k-CF-coloring) *Let $H = (V, E)$ be a hypergraph, k be a positive integer. A coloring $\chi : V \to \{1, \ldots, m\}$ is called a k-CF-coloring of H if for any $e \in \mathcal{E}$ there is a color j such that $1 \leq |\{v \in e | \chi(v) = j\}| \leq k$.*

Let $\chi_{kcf}(H)$ denote the smallest number of colors in any possible k-CF-coloring of H. Note that 1-CF-coloring of a hypergraph is simply a CF-coloring.

Refer to Smorodinsky [19] and Har-Peled and Smorodinsky [14] for the study of k-CF-coloring. Another extension of CF-coloring is called k-SCF-coloring, which is defined as follows:

Definition 1.3 (k-SCF-coloring) *Let $H = (V, \mathcal{E})$ be a hypergraph, k be a positive integer. A coloring $\chi : V \to \{1, \ldots, m\}$ is called a k-SCF-coloring if for any $e \in \mathcal{E}$ with $|e| \geq k$, there are at least k colors which appear only once in e, and for any $e \in \mathcal{E}$ with $|e| < k$ all points in e are of different colors.*

Let $f_H(k)$ denote the smallest number of colors in any possible k-SCF-coloring of H. Note that 1-SCF-coloring is just a CF-coloring.

Abellanas et al. [1] were the first to study k-SCF-coloring[2]. Aloupis et al. [4] introduced another coloring called k-colorful coloring, which has interesting connection with strong-conflict-free coloring. Refer to Horev et al. [15] for the connection and research for k-SCF-coloring.

Throughout the rest of this paper, we let $H_n = (V_n, \mathcal{E}_n)$ be a hypergraph, where $V_n = \{1, 2, \ldots, n\}$, and $\mathcal{E}_n = \{E \subset V_n | E = (a, b) \cap V_n, a, b \in \mathbb{R}, a < b \text{ and } E \neq \emptyset\}$. Har-Peled and Smorodinsky [14] proves that $\chi_{cf}(H_n) = \lfloor \log n \rfloor + 1$ as a simple yet an important example of CF-coloring of a hypergraph.

In Section 2, we consider k-SCF-coloring of H_n and give the exact k-SCF-coloring number of H_n for $k = 2, 3$. In Section 3, we give the exact k-CF-coloring number of H_n for all k. In the last section, we consider online k-CF-coloring of any hypergraph.

For simplicity, in the following we denote by $\lceil x \rceil_e$ the least even number greater than x, and $\lceil x \rceil_o$ the least odd number greater than x for $x \geq 0$.

---

[2]They referred to such a coloring as k-conflict-free coloring

2 k-SCF-coloring of H_n

In this section, we consider k-SCF-coloring of H_n and focus on $f_{H_n}(k)$ especially. To this end, define for $m = 1, 2, \ldots,$

$$g_k(m) = \sup\{n : f_{H_n}(k) \leq m\},$$

i.e. $g_k(m)$ is the largest n such that we can give H_n a k-SCF-coloring by using m colors. The idea is that if we can get one clear expression of $g_k(m)$ as a function of m, then we will be able to obtain $f_{H_n}(k)$ by the formula $f_{H_n}(k) = \inf\{m : g_k(m) \geq n\}$. Generally, we have the following inequalities.

Theorem 2.1 *Suppose* $k, m, p \in \mathbb{N}$, $m \geq k$ *and* $m > p + 1$. *Then we have*
(i) if $k = 2p$, *then* $g_k(m) \leq g_k(m - p) + g_k(m - p - 1) + 1$;
(ii) if $k = 2p + 1$, *then* $g_k(m) \leq 2g_k(m - p - 1) + 1$.

Proof. (i) Suppose that $k = 2p$ and the inequality in (i) is not true, then there is some way to color $g_k(m - p) + g_k(m - p - 1) + 2$ points using m colors and the coloring is k-SCF. Suppose these $g_k(m - p) + g_k(m - p - 1) + 2$ points are

$$\underbrace{1,\ 2,\ \ldots,\ g_k(m-p)-1,\ g_k(m-p),}_{M\ region}\ A,\ \underbrace{g_k(m-p)+2,\ g_k(m-p)+3,\ \ldots,\ g_k(m-p)+g_k(m-p-1)+1,}_{N\ region}\ B,$$

where A, B are two points. Because the coloring is k-SCF, there are k colors $\{a_1, \ldots, a_k\}$, which appear only once over the $g_k(m - p) + g_k(m - p - 1) + 2$ points. But no more than p (including p) of these k colors could appear in $\{N\ region\} \cup \{B\}$, otherwise there will be at most $m - p$ colors in $\{M\ region\} \cup \{A\}$, which is a contradiction with respect to (w.r.t. for short) the definition of $g_k(m-p)$. So there are at most $p - 1$ colors of $\{a_1, \ldots, a_k\}$ which appear in $\{N\ region\} \cup \{B\}$, that is, at least $k - (p - 1) = p + 1$ colors of $\{a_1, \ldots, a_k\}$ appear in $\{M\ region\} \cup \{A\}$. So now there will be at most $m - (p + 1) = m - p - 1$ colors which appear in $\{N\ region\} \cup \{B\}$. This is a contradiction w.r.t. the definition of $g_k(m - p - 1)$. Hence we have that $g_k(m) \leq g_k(m - p) + g_k(m - p - 1) + 1$.

(ii) Suppose that $k = 2p + 1$ and the inequality in (ii) is not true. Then there is some way to color $2g_k(m - p - 1) + 2$ points using m colors and the coloring is k-SCF. Let these $2g_k(m - p - 1) + 2$ points be arranged as follows:

$$\underbrace{1,\ 2,\ \ldots,\ g_k(m-p-1)-1,\ g_k(m-p-1),}_{M\ region}\ A,\ \underbrace{g_k(m-p-1)+2,\ g_k(m-p-1)+3,\ \ldots,\ 2g_k(m-p-1)+1,}_{N\ region}\ B.$$

By k-SCF-coloring condition, there are k colors $\{a_1, \ldots, a_k\}$, which appear only once over the $2g_k(m - p - 1) + 2$ points. By the same arguments in (i) above, we know that there are at most p colors of $\{a_1, \ldots, a_k\}$ which appear in $\{N\ region\} \cup \{B\}$, and thus at least $k - p = p + 1$ colors of $\{a_1, \ldots, a_k\}$ appear in $\{M\ region\} \cup \{A\}$. So now there will be at most $m - (p + 1) = m - p - 1$ colors which appear in $\{N\ region\} \cup \{B\}$. This is a contradiction w.r.t. the definition of $g_k(m - p - 1)$. Hence we have $g_k(m) \leq 2g_k(m - p - 1) + 1$. $\qquad\square$

Now we focus on two simple cases that $k = 2$ and $k = 3$.

Theorem 2.2 *For any $m = 2, 3, \ldots$, we have*

$$g_2(m+1) = g_2(m) + g_2(m-1) + 1. \tag{2.1}$$

Proof. By Theorem 2.1(i), in order to prove (2.1), we need only prove that for any $m = 2, 3, \ldots$, $g_2(m+1) \geq g_2(m) + g_2(m-1) + 1$, i.e. there exists a 2-SCF-coloring for $g_2(m) + g_2(m-1) + 1$ points by using $m+1$ colors.

One can check that $g_2(1) = 1$, $g_2(2) = 2$. Then we can obtain that $g_2(3) = 4$, $g_2(4) = 7$, $g_2(5) = 12, \ldots, g_2(m+1) = g_2(m) + g_2(m-1) + 1$, if we can give a 2-SCF-coloring to 4 points by using 3 colors, to 7 points by using 4 colors, to 12 points by using 5 colors, $\ldots$, to $g_2(m) + g_2(m-1) + 1$ points by using $m+1$ colors. For simplicity, in the following we use one sequence $a_1, a_2, \ldots, a_n$, to denote a 2-SCF-coloring $\chi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ for some m with the meaning that $\chi(i) = a_i$ for $i = 1, \ldots, n$.

When $m = 1, 2, 3$ and 4, we have the following 2-SCF-coloring respectively:

- **1**;

- **1, 2**;

- $1, \mathbf{2}, \mathbf{3}, 1$;

- $1, 2, \mathbf{3}, 1, \mathbf{4}, 2, 1$.

When $m = 5$, we have the following 2-SCF-coloring

$$1, 2, 3, 1, \mathbf{4}, 2, 1, \mathbf{5}, 2, 3, 1, 2.$$

Note that the color sequence on the left side of "5" is "$1, 2, 3, 1, 4, 2, 1$". The reversion of the sequence on the right side of "4" is "$2, 1, 3, 2, 5, 1, 2$". One of these two color sequences is changed to the other one if we exchange the colors 5 and 4, 2 and 1.

When $m = 6$, the 2-SCF-coloring is

$$1, 2, 3, 1, 4, 2, 1, \mathbf{5}, 2, 3, 1, 2, \mathbf{6}, 3, 2, 4, 3, 1, 2, 3.$$

Note that the color sequence on the left side of "6" is "$1, 2, 3, 1, 4, 2, 1, 5, 2, 3, 1, 2$". The reversion of the sequence on the right side of "5" is "$3, 2, 1, 3, 4, 2, 3, 6, 2, 1, 3, 2$". One of these two color sequences is changed to the other one if we exchange the colors 6 and 5, 3 and 1.

Now we assume that for $m \geq 3$ and any $l = 3, \ldots, m$, we have constructed the 2-SCF-coloring to $g_2(l) + g_2(l-1) + 1$ points by using $l + 1$ colors and following the above idea. Denote the 2-SCF-coloring for $l = m$ by the following color sequence

$$1, 2, \ldots, a_{g_2(m-2)}, \mathbf{m-1}, a_{g_2(m-2)+2}, \ldots, a_{g_2(m-1)}, \mathbf{m}, a_{g_2(m-1)+2}, \ldots, a_{g_2(m)}, \tag{2.2}$$

where the colors $\mathbf{m-1}$ and $\mathbf{m}$ appear only once. Basing on the coloring (2.2), we construct one $(m+1)$-coloring for $g_2(m) + g_2(m-1) + 1$ points as follows:

$$\begin{aligned}
&1, 2, \ldots, a_{g_2(m-2)}, m-1, a_{g_2(m-2)+2}, \ldots, a_{g_2(m-1)}, \mathbf{m}, a_{g_2(m-1)+2}, \ldots, a_{g_2(m)}, \mathbf{m+1}, \\
&\overline{a_{g_2(m-1)}}, \ldots, \overline{a_{g_2(m-2)+2}}, \overline{m-1}, \overline{a_{g_2(m-2)}}, \ldots, \overline{2}, \overline{1},
\end{aligned} \tag{2.3}$$

4

where the color sequence on the left side of "**m+1**"

$$1, 2, \ldots, a_{g_2(m-2)}, m-1, a_{g_2(m-2)+2}, \ldots, a_{g_2(m-1)}, \mathbf{m}, a_{g_2(m-1)+2}, \ldots, a_{g_2(m)} \tag{2.4}$$

and the reversion of the sequence on the right side of "**m**"

$$\bar{1}, \bar{2}, \ldots, \overline{a_{g_2(m-2)}}, \overline{m-1}, \overline{a_{g_2(m-2)+2}}, \ldots, \overline{a_{g_2(m-1)}}, \mathbf{m+1}, a_{g_2(m)}, \ldots, a_{g_2(m-1)+2} \tag{2.5}$$

can be transformed mutually if we exchange the colors $\mathbf{m+1}$ and $\mathbf{m}$, and some other color pairs which appear in the color sequence "$a_{g_2(m-1)+2}, \ldots, a_{g_2(m)}$" between the colors $\mathbf{m}$ and $\mathbf{m+1}$. Obviously, the colors $\mathbf{m}$ and $\mathbf{m+1}$ appear only once.

Next we prove that the coloring (2.3) is 2-SCF. For any hyperedge e of $H_{g_2(m)+g_2(m-1)+1}$, if it contains both color $\mathbf{m}$ and color $\mathbf{m+1}$, then it satisfies the condition of 2-SCF-coloring. If not, then the color sequence associated with e is a subsequence of (2.4) or (2.5), and thus it satisfies the condition of 2-SCF-coloring by our construction. □

Corollary 2.3 *For any $n = 1, 2, \ldots$, we have $f_{H_n}(2) = \min\{A_n, B_n\}$, where*

$$A_n = \left\lceil \log_{\frac{1+\sqrt{5}}{2}} \frac{\sqrt{5}(n+1) + \sqrt{5(n+1)^2 - 4}}{2(2 + \sqrt{5})} + 1 \right\rceil_o,$$

$$B_n = \left\lceil \log_{\frac{1+\sqrt{5}}{2}} \frac{\sqrt{5}(n+1) + \sqrt{5(n+1)^2 + 4}}{2(2 + \sqrt{5})} + 1 \right\rceil_e.$$

Proof. By Theorem 2.2, we have $g_2(m+1) = g_2(m) + g_2(m-1) + 1$, $\forall m \geq 2$. Let $\hat{g}_2(m) = g_2(m) + 1$, $\forall m \geq 2$. Then $\hat{g}_2(m)$ satisfies the following recursive relation:

$$\hat{g}_2(m+1) - \hat{g}_2(m) - \hat{g}_2(m-1) = 0.$$

By the theory of linear sequence, the character equation of $\hat{g}_2(m)$ is

$$\lambda^2 - \lambda - 1 = 0,$$

with two roots being $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha}$. So we can write

$$\hat{g}_2(m) = C_1 \alpha^m + C_2 \beta^m, \quad C_1, C_2 \in \mathbb{R}.$$

By $\hat{g}_2(1) = 2$ and $\hat{g}_2(2) = 3$, we get that

$$C_1 = \frac{3 - 2\beta}{\alpha(\alpha - \beta)}, \quad C_2 = \frac{2\alpha - 3}{\beta(\alpha - \beta)}.$$

Thus we have

$$
\begin{aligned}
g_2(m) &= \hat{g}_2(m) - 1 \\
&= \frac{(3 - 2\beta)\alpha^{m-1} + (2\alpha - 3)\beta^{m-1}}{\alpha - \beta} - 1 \\
&= \frac{1}{\sqrt{5}} \left[(2 + \sqrt{5}) \left(\frac{1+\sqrt{5}}{2} \right)^{m-1} + (\sqrt{5} - 2)(-1)^{m-1} \left(\frac{1+\sqrt{5}}{2} \right)^{-(m-1)} \right] - 1. \tag{2.6}
\end{aligned}
$$

By (2.6) and the formula $f_{H_n}(2) = \inf\{m : g_2(m) \geq n\}$, we obtain that $f_{H_n}(2) = \min\{A_n, B_n\}$. □

Theorem 2.4 *For any $m = 3, 4, \ldots$, we have*

$$g_3(m) = 2g_3(m-2) + 1. \tag{2.7}$$

Proof. By Theorem 2.1(ii), in order to prove (2.7), we need only prove that for any $m = 3, 4, \ldots$, $g_3(m) \geq 2g_3(m-2) + 1$, i.e. there exists a 3-SCF-coloring for $2g_3(m-2) + 1$ points by using m colors. As in the proof of Theorem 2.2, in the following we use one sequence $a_1, a_2, \ldots, a_n$, to denote a 3-SCF-coloring $\chi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ for some m with the meaning that $\chi(i) = a_i$ for $i = 1, \ldots, n$.

Step 1. We consider the sequence $\{3, 5, 7, \ldots\}$ of odd numbers. For $m = 3, 5, 7$, we have the following 3-SCF-colorings:

- $1, 2, 3$;

- $1, 2, 3, 4, 1, 5, 3$;

- $1, 2, 3, 4, 1, 5, 3, 6, 1, 2, 3, 7, 1, 5, 3$.

Hence $g_3(3) = 3 = 2g_3(1) + 1$; $g_3(5) = 7 = 2g_3(3) + 1$; $g_3(7) = 15 = 2g_3(5) + 1$.

Note that the third coloring for $m = 7$ has the following structure:

$$\underbrace{1,2,3,}_{A} \; \mathbf{4}, \; \underbrace{1,5,3,}_{B} \; \mathbf{6}, \; \underbrace{1,2,3,}_{A} \; \mathbf{7}, \; \underbrace{1,5,3}_{B} \; . \tag{2.8}$$

When $m = 9$, two new colors 8 and 9 are added. We can construct the following coloring:

$$A, 4, B, \mathbf{6}, A, 7, B, \mathbf{8}, A, 4, B, \mathbf{9}, A, 7, B. \tag{2.9}$$

Now we show that (2.9) is a 3-SCF-coloring. Notice that the three colors 6, 8, 9 appear only once.

If a hyperedge contains the colors 6, 8 and 9, then it satisfies the condition of 3-SCF-coloring. So we need only check those hyperedges which do not contain all these three colors. Hyperedges which do not contain all the colors 6, 8 and 9 have the following four types (with overlapping):

1. Those which do not contain color 9;

2. Those which do not contain color 6;

3. Those which do not contain color 8 and color 9;

4. Those which do not contain color 6 and color 8.

Type 4 is a version of type 3 with color 6 being substituted by color 9, so any hyperedge belongs to these two types satisfies the condition of 3-SCF-coloring by the 3-SCF-coloring property of (2.8). Type 2 is a version of type 1 with colors 4, 6, 7, 8 being substituted by 7, 8, 4, 9, respectively. So we need only check type 1.

6

Notice that colors 6, 7, and 8 appear only once in the color sequence "$A, 4, B, \mathbf{6}, A, 7, B, \mathbf{8}, A, 4, B$" on the left side of color 9 in (2.9). Then any hyperedge of type 1 which contains colors 6, 7 and 8 satisfies the condition of 3-SCF-coloring. Further, those of type 1 which do not contain all the colors 6, 7 and 8 have the following types (with overlapping):

5. Those which do not contain color 8;

6. Those which do not contain color 6;

7. Those which do not contain color 7 and color 8;

8. Those which do not contain color 6 and color 7.

Type 6 is a version of type 5 with colors 4, 6, 7 being substituted by colors 7, 8, 4, respectively. Type 5 belongs to type 3 and so any hyperedge in types 5 and 6 satisfies the condition of 3-SCF-coloring. Type 8 is a version of type 7 with A and B being exchanged, and colors 4, 6 being substituted by colors 8, 4. Any hyperedge of type 7 satisfies the condition of 3-SCF-coloring by the 3-SCF-coloring for $m = 7$.

In a word, the coloring (2.9) with $m = 9$ colors is 3-SCF. Hence $g_3(9) = 2g_3(7) + 1$.

Next we show that the coloring (2.9) still has the structure of (2.8):

$$\underbrace{A, 4, B}_{A'}, 6, \underbrace{A, 7, B}_{B'}, 8, \underbrace{A, 4, B}_{A'}, 9, \underbrace{A, 7, B}_{B'}, \tag{2.10}$$

The structure is reserved by substituting A and B by A' and B', and the colors 6, 8, 9 appears only once in the whole coloring (2.10), just as 4, 6, 7 do in coloring (2.8).

Because the coloring structure is reserved, new 3-SCF-coloring could always be constructed when two new colors are added. Hence by induction, we get that for $m = 3, 5, 7, \ldots$, (2.7) holds.

Step 2. We consider the sequence $\{4, 6, 8, \ldots\}$ of even numbers. For $m = 4$, we have the following 3-SCF-coloring

$$1, 2, 3, 4, 1.$$

So $g_3(4) = 2g_3(2) + 1 = 5$.

For $m = 6$, we have the following 3-SCF-coloring

$$1, 2, 3, \mathbf{4}, 1, 5, 3, \mathbf{6}, 1, 2, 3,$$

which can be expressed by

$$A_3, \mathbf{4}, B_3, \mathbf{6}, A_3, \tag{2.11}$$

where $A_3 = \{1, 2, 3\}$ and $B_3 = \{1, 5, 3\}$. So $g_3(6) = 2g_3(4) + 1 = 11$.

For $m = 8$, we have the following 3-SCF-coloring

$$\underbrace{1,2,3,}_{A_3} 4, \underbrace{1,5,3,}_{B_3} \mathbf{6}, \underbrace{1,2,3,}_{A_3} \mathbf{7}, \underbrace{1,5,3}_{B_3} \mathbf{8}, \underbrace{1,2,3,}_{A_3} 4, \underbrace{1,5,3,}_{B_3},$$

7

which can be expressed by

$$A_4, \mathbf{6}, B_4, \mathbf{8}, A_4, \tag{2.12}$$

where $A_4 = \{A_3, 4, B_3\}$ and $B_4 = \{A_3, 7, B_3\}$. So $g_3(8) = 2g_3(6) + 1 = 23$.

For $m = 10$, we can construct the following 3-SCF-coloring

$$A_4, 6, B_4, \mathbf{8}, A_4, \mathbf{9}, B_4, \mathbf{10}, A_4, 6, B_4,$$

which can be expressed by

$$A_5, \mathbf{8}, B_5, \mathbf{10}, A_5, \tag{2.13}$$

where $A_5 = \{A_4, 6, B_4\}$ and $B_5 = \{A_4, 9, B_4\}$. So $g_3(10) = 2g_3(8) + 1 = 47$.

Notice that (2.11), (2.12) and (2.13) have the same structure, and we can construct 3-SCF-coloring for $m = 2(k+1)$ basing the coloring with this structure for $m = 2k, k = 3, 4, \ldots$. Hence for $m = 4, 6, 8, \ldots$, (2.7) holds. The proof is completed. $\qquad\square$

Corollary 2.5 *For any $n = 1, 2, \ldots$, we have*

$$f_{H_n}(3) = \min \left\{ \left\lceil 2(1 + \log_2 \frac{\mathrm{n}+1}{3}) \right\rceil_{\mathrm{e}}, \lceil 2\log_2(\mathrm{n}+1) + 1 \rceil_{\mathrm{o}} \right\}. \tag{2.14}$$

Proof. By Theorem 2.4, we have known that $g_3(m) = 2g_3(m-2) + 1$, $\forall m \geq 3$. Let $\hat{g}_3(m) = g_3(m) + 1$, $\forall m \geq 3$, then $\hat{g}_3(m)$ satisfies the following recursive relation

$$\hat{g}_3(m+2) = 2\hat{g}_3(m), \ \forall m \geq 1,$$

which together with $\hat{g}_3(1) = 2, \hat{g}_3(2) = 3$ implies that

$$g_3(m) = \begin{cases} 3 \cdot 2^{p-1} - 1 & \text{if } m = 2p ; \\ 2^p - 1 & \text{if } m = 2p + 1. \end{cases} \tag{2.15}$$

By (2.15) and the formula $f_{H_n}(3) = \inf\{m : g_3(m) \geq n\}$, we obtain (2.14). $\qquad\square$

3 k-CF-coloring of H_n

In this section, we consider k-CF-coloring of H_n for any $k = 1, 2, \ldots$ and obtain the following result.

Theorem 3.1 *For any $k, n = 1, 2, \ldots$, we have $\chi_{kCF}(H_n) = \lfloor log_{(k+1)}n \rfloor + 1$.*

Proof. Step 1: We prove that for any $m = 1, 2, \ldots$, when $n \geq (k+1)^m$, $\chi_{kCF}(H_n) \geq m + 1$. If $m = 1$, it is right. Suppose that the claim holds for some $m \in \{1, 2, \ldots\}$. For $n \geq (k+1)^{m+1}$, we express $V_n = \{1, 2, \ldots, n\}$ by

$$V_n = V_{n,1} \cup V_{n,2} \cup \cdots \cup V_{n,k+1} \cup V_{n,k+2}, \tag{3.1}$$

where

$$V_{n,1} = \{1, 2, \ldots, (k+1)^m\},$$
$$V_{n,2} = \{(k+1)^m + 1, (k+1)^m + 2, \ldots, 2(k+1)^m\},$$
$$\cdots$$
$$V_{n,k+1} = \{k(k+1)^m + 1, k(k+1)^m + 2, \ldots, (k+1)^{m+1}\},$$

and if $n = (k+1)^{m+1}$, then $V_{n,k+2} = \emptyset$; if $n > (k+1)^{m+1}$, then $V_{n,k+2} = \{(k+1)^{m+1} + 1, \ldots, n\}$. By the expression (3.1) and the induction hypothesis, we can easily get that $\chi_{kCF}(H_n) \geq m + 2$. Hence the claim holds for any $m = 1, 2, \ldots$.

Step 2: We prove that for $m = 1, 2, \ldots, n = (k+1)^m - 1$, $\chi_{kCF}(H_n) = m$. When $m = 1$, $n = k$ and so $\chi_{kCF}(H_n) = 1 = m$. Suppose that the claim holds for some $m \in \{1, 2, \ldots\}$. For $n = (k+1)^{m+1} - 1$, we express $V_n = \{1, 2, \ldots, n\}$ by

$$V_n = \overline{V}_{n,1} \cup \{(k+1)^m\} \cup \overline{V}_{n,2} \cup \{2(k+1)^m\} \cup \cdots \cup \overline{V}_{n,k} \cup \{k(k+1)^m\} \cup \overline{V}_{n,k+1}, \tag{3.2}$$

where

$$\overline{V}_{n,1} = \{1, 2, \ldots, (k+1)^m - 1\},$$
$$\overline{V}_{n,2} = \{(k+1)^m + 1, (k+1)^m + 2, \ldots, 2(k+1)^m - 1\},$$
$$\cdots$$
$$\overline{V}_{n,k+1} = \{k(k+1)^m + 1, k(k+1)^m + 2, \ldots, (k+1)^{m+1} - 1\}.$$

For any $i = 1, \ldots, k + 1$, by the induction hypothesis, we know that the sub-hypergraph induced by $\overline{V}_{n,i}$ has a k-CF-coloring by using m colors e.g. colors $1, 2, \ldots, m$. Use color $m + 1$ to color the vertices $(k+1)^m, 2(k+1)^m, \ldots, k(k+1)^m$. We can easily check that this coloring is a k-CF-coloring of H_n. So $\chi_{kCF}(H_n) \leq m + 1$. By **Step 1** and the fact that $(k+1)^{m+1} - 1 \geq (k+1)^m$, we get that $\chi_{kCF}(H_n) = m + 1$ for $n = (k+1)^{m+1} - 1$. Hence for any $m = 1, 2, \ldots$, the claim holds.

By **Step 1** and **Step 2**, we obtain that for any $k, n = 1, 2, \ldots, \chi_{kCF}(H_n) = \lfloor log_{(k+1)} n \rfloor + 1$. $\square$

4 Online k-CF-coloring of hypergraphs

To capture a dynamic scenario where antennae can be added to the network, Chen et al. [8] initiated the study of online conflict-free colouring of hypergraphs. They proposed a natural, simple, and obvious coloring algorithm called the UniMax greedy algorithm, but showed that the UniMax greedy algorithm may require $\Omega(\sqrt{n})$ colors in the worst case. They also introduced

a 2-stage deterministic variant of the UniMax greedy algorithm and showed that the maximum number of colors that it uses is $\Theta(\log^2 n)$. In addition, they described a randomized version of the UniMax greedy algorithm, which uses, with high probability, only $O(\log n)$ colors.

Among other results, Fiat et al. [13] provided a randomized algorithm for online conflict-free coloring of n points on the line with $O(\log n \log \log n)$ colors with high probability. Chen, Kaplan and Sharir [7, 9] considered the hypergraphs induced by points in the plane with respect to intervals, half-planes, and unit disks and otained randomized online conflict-free coloring algorithm that use $O(\log n)$ colors with high probability.

Bar-Noy et al. [5] gave a more general framework for online CF-coloring of hypergraph than [7, 8, 9]. This framework is used to obtain efficient randomized online algorithms for hypergraphs provided that a special parameter referred to as the degeneracy of the underlying hypergraph is small.

In this section, we extend some results about the online conflict-free coloring in [5] to online k-CF-coloring. First, we give some necessary definitions. Second, we present a framework for online k-CF-coloring. Finally, we give an online randomized k-CF-coloring algorithm.

4.1   Some definitions

Definition 4.1 *For a hypergraph $H = (V, E)$ and an integer $m \geq 2$, define an m-uniform hypergraph $D_m(H) = (V, F)$, where $F = \{e \in E \,|\, |e| = m\}$.*

If $m = 2$, then $D_m(H)$ is just the *Delaunay graph* $G(H)$ of H.

Definition 4.2 *Let $H = (V, \mathcal{E})$ be a hypergraph and ϕ be an m-coloring of H. If for any $e \in \mathcal{E}$ with $|e| \geq k + 1$, we have*
$$|\{a \mid \ \exists v \in e \ \ s.t. \ \phi(v) = a\}| \geq 2,$$
then ϕ is called k-proper non-monochromatic.

Notice that 1-*proper non-monochromatic* coloring is just *proper* or *non-monochromatic* coloring.

Definition 4.3 *Let $k > 0, q > 0$ be two fixed integers and $H = (V, E)$ be a hypergraph on the n vertices $v_1, v_2, \ldots, v_n$. For a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, and $t = 1, \ldots, n$, define*
$$S_k^\pi(t) = \sum_{j=1}^{t} d_k(v_{\pi(j)}),$$
where $d_k(v_{\pi(j)}) = |\{e \in D_{k+1}(H(\{v_{\pi(1)}, \ldots, v_{\pi(j)}\})) : v_{\pi(j)} \in e\}|$. If, for any permutation π and any $t \in \{1, 2, \ldots, n\}$, we have
$$S_k^\pi(t) \leq qt,$$
then we say that H is q-degenerate of degree k.

Notice that q-degenerate of degree 1 is just q-degenerate defined in [5].

4.2 A framework for online k-CF-coloring

Let $H = \{V, \mathcal{E}\}$ be any hypergraph. We define a framework that colors the vertices of V in an online fashion, i.e., when the vertices of V are revealed by an adversary one at a time. At each time step t, the algorithm must assign a color to the newly revealed vertex v_t. This color cannot be changed in future times $t' > t$. The coloring has to be k-conflict-free for all the induced hypergraphs $H(V_t)$ with $t = 1, ..., n$, where $V_t \subseteq V$ is the set of vertices revealed by time t. The framework is almost the same with the one for online conflict-free coloring in [5]. In fact, we need only change "proper non-monochromatic coloring" to "k-proper non-monochromatic coloring". For the reader's convenience, we spell out the details.

For a fixed positive integer h, let $A = \{a_1, ..., a_h\}$ be a set of h auxiliary colors. Let $f : \mathbb{N} \to A$ be some fixed function. In the following, we define the framework that depends on the choice of the function f and the parameter h.

A table (to be updated online) is maintained with row entries indexed by the variable i with range in $\mathbb{N}$. Each row entry i at time t is associated with a subset $V_t^i \subseteq V_t$ in addition to an auxiliary k-proper non-monochromatic coloring of $H(V_t^i)$ with at most h colors. We say that $f(i)$ is the auxiliary color that represents entry i in the table. At the beginning all entries of the table are empty. Suppose all entries of the table are updated until time $t - 1$ and let v_t be the vertex revealed by the adversary at time t. The framework first checks if an auxiliary color can be assigned to v_t such that the auxiliary coloring of V_{t-1}^1 together with the color of v_t is a k-proper non-monochromatic coloring of $H(V_{t-1}^1 \cup \{v_t\})$. Any ($k$-proper non-monochromatic) coloring procedure can be used by the framework: for example, a first-fit greedy method in which all colors in the order $a_1, ..., a_h$ are checked until one is found. If such a color cannot be found for v_t, then entry 1 is left with no changes and the process continues to the next entry. If however, such a color can be assigned, then v_t is added to the set V_{t-1}^1. Let c denote such an auxiliary color assigned to v_t. If this color is the same as $f(1)$ (the auxiliary color that represents entry 1), then the final color of v_t in the online k-CF-coloring is 1 and the updating process for the t-th vertex stops. Otherwise, if an auxiliary color cannot be found or if the assigned auxiliary color is not the same as $f(1)$, then the updating process continues to the next entry. The updating process stops at the first entry i for which v_t is both added to V_t^i and the auxiliary color assigned to v_t is the same as $f(i)$. Then, the main color of v_t in the final k-CF-coloring is set to i.

It is possible that v_t never gets a final color. In this case we say that the framework does not halt. However, termination can be guaranteed by imposing some restrictions on the auxiliary coloring method and the choice of the function f. Later, a randomized online algorithm based on this framework is derived under the oblivious adversary model. This algorithm always halts, or to be more precise halts with probability 1, and moreover it halts after a "small" number of entries with high probability. We prove that the above framework produces a valid k-CF-coloring if it halts.

Proposition 4.4 *If the above framework halts for any vertex v_t then it produces a valid online k-CF-coloring of H.*

Proof. The proof is similar to the one of [5, Lemma 3.1]. Let $H(V_t)$ be the hypergraph induced

by the vertices already revealed at time t. Let S be a hyperedge in this hypergraph and let j be the maximum integer for which there is a vertex v of S colored with j. We claim that at most k such vertices in S exist. Assume to the contrary that there is (for example) $k+1$ vertices in S colored with j. This means that at time t all these $k+1$ vertices were present at entry j of the table and that they all got an auxiliary color (in the auxiliary coloring of the set V_t^j) which equals $f(j)$. However, since the auxiliary coloring is a k-proper non-monochromatic coloring of the induced hypergraph at entry j, $S \cap V_t^j$ is not monochromatic so there must exist a $(k+2)$-th vertex $v' \in S \cap V_t^j$ that was present at entry j and was assigned an auxiliary color different from $f(j)$. Thus, v' got its final color in an entry greater than j, which contradicts the maximality of j in the hyperedge S. $\qquad\square$

4.3   An online randomized k-CF-coloring algorithm

Based on the framework of Section 4.2, we can obtain an online randomized k-CF-coloring algorithm by using the same choices with [5] for (a) the set of auxiliary colors of each entry, (b) the function f, and (c) the algorithm for the auxiliary coloring at each entry, i.e. we use the set of auxiliary colors $A = \{a_1, ..., a_{2q+1}\}$, where q is a parameter on degeneracy of the hypergraph, see Theorem 4.5 below; for each entry i, the representing color $f(i)$ is chosen uniformly at random from A; we use a first-fit algorithm for the auxiliary coloring.

Following the proof of [5, Theorem 4.1], we obtain the following result. The complete proof can be found in [11].

Theorem 4.5 *Let $H = (V, \mathcal{E})$ be a q-degenerate hypergraph of degree k on n vertices. Then, there exists a randomized online k-CF-coloring algorithm for H which uses at most $O(\log_{1+\frac{1}{4q+1}} n) = O(q\log n)$ colors with high probability against an oblivious adversary.*

References

[1] Abellanas, M., Bose, P., García, J., Hurtado, F., Nicolás, M. and Ramos, P.A. (2005) On properties of higher order delaunay graphs with applications. In *21st European Workshop on Computational Geometry* (EWCG), pp. 119-122.

[2] Ajwani, D., Elbassioni, K., Govindarajan, S. and Ray, S. (2007) Conflict-free coloring for rectangle ranges using $O(n^{.382})$ colors. In *Proc. 19th Annual ACM Symposium on Parallelism in Algorithms and Architectures* (SPAA), pp. 181-187.

[3] Alon, N. and Smorodinsky, S. (2008) Conflict-free coloring of shallow discs. *Int. J. Comput. Geometry Appl.* **18** 599-604.

[4] Aloupis, G., Cardinal, J., Collette, S., Langerman S. and Smorodinsky, S. (2008) Coloring geometric range spaces. In *Proc. of the 8th Latin American Symposium on Theoretical Informatics (LATIN'08)*, pp. 146-157.

[5] Bar-Noy, A., Cheilaris, P., Olonetsky, S. and Smorodinsky, S. (2010) Online conflict-free coloring for hypergraphs. *Comb. Probab. Comput.* **19** 493-516.

[6] Bar-Noy, A., Cheilaris, P. and Smorodinsky, S. (2006) Conflict-free colorings for intervals: from offline to online. In *Proc. 18th Annual ACM Symposium on Parallelism in Algorithms and Architectures* (SPAA), pp. 128-137.

[7] Chen, K. (2006) How to play coloring game against a color-blind adversary. In *Proc. 22nd Annual ACM Symposiu on Computational Geometry* (SoCG), pp. 44-51.

[8] Chen, K., Fiat, A., Kaplan, H., Levy, M., Matoušek, J., Mossel E., Pach J., Sharir, M., Smorodinsky S., Wagner, U. and Welzl, E. (2007) Online conflict-free coloring for intervals. *SIAM J. Comput.* **36** 1342-1359.

[9] Chen, K., Kaplan, H. and Sharir, M. (2009) Online conflict-free coloring for halfplanes, congruent disks, and axis-parallel rectangles. *ACM Trans. on Algorithm* **5** 16:1-16:24.

[10] Chen, X., Pach J., Szegedy, M. and Tardos, G. (2008) Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles. *Random Structures and Algorithms* **34** 11-23.

[11] Cui, Z. (2011) Conflict-free coloring and related problems. Master thesis, Department of Mathematics, Nanjing University.

[12] Even, G., Lotker, Z., Ron, D. and Smorodinsky, S. (2003) Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comput.* **33** 94-136.

[13] Fiat, A., Lotker, Z., Ron, D., Mossel, E., Pach, J., Smorodinsky, S., Wagner, U. and Welzl, E. (2005) Online conflict-free coloring for intervals. In *Proc. 16th Annual ACM-SIAM Symposium on Discrete Algorithms* (SODA), pp. 545-554.

[14] Har-Peled, S. and Smorodinsky, S. (2005) Conflict-free coloring of points and simple regions in the plane. *Discrete and Computat. Geometry* **34** 47-70.

[15] Horev, E., Krakovski, R. and Smorodinsky, S. (2010) Conflict-free coloring made stronger. *Lecture Notes in Computer Science, Volume 6139*, pp. 105-117.

[16] Lev-Tov, N. and Peleg, D. (2009) Conflict-free coloring of unit disks. *Discrete Appl. Math.* **157** 1521-1532.

[17] Pach, J. and Tardos, G. (2009) Conflict-free colorings of graphs and hypergraphs. *Comb. Probab. Comput.* **18** 819-834.

[18] Pach, J. and tardos, G. (2010) Coloring axis-parallel rectangles. *J. Comb. Theory, Series A* **117** 776-782.

[19] Smorodinsky, S. (2003) Combinatorial Problems in Computational Geometry. PhD thesis, School of Computer Science, Tel-Aviv University.

[20] Smorodinsky, S. (2007) On the chromatic number of some geometric hypergraphs. *SIAM J. Discrete Math.* **21** 676-687.

[21] Smorodinsky, S. (2011) Conflict-free coloring and its applications. arXiv:1005.3616v2 [math.CO].